الجامعة السورية الخاصة
SYRIAN PRIVATE UNIVERSITY

كـلـيــة هندسة الحاسوب والمعلوماتية
Computer and Informatics Engineering Faculty

Electric Circuits I

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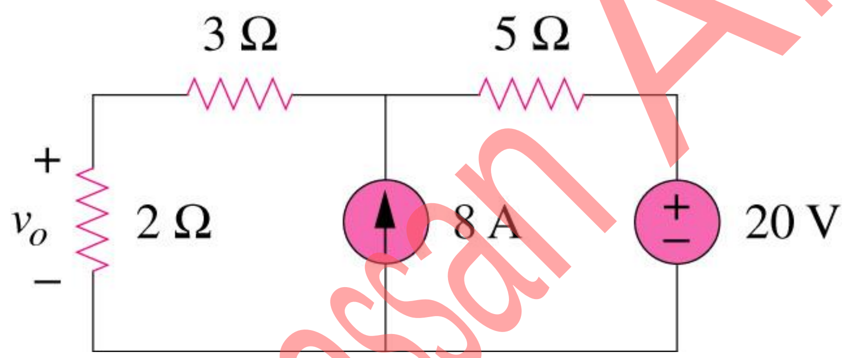
Chapter 4
Circuit Theorems

4.1 Motivation
4.2 Linearity Property
4.3 Superposition
4.4 Source Transformation
4.5 Thevenin's Theorem
4.6 Norton's Theorem
4.7 Maximum Power Transfer

4.1 Motivation

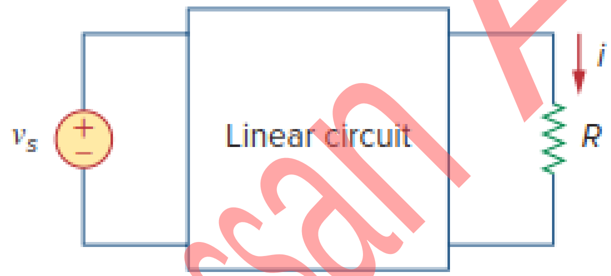
If you are given the following circuit, are there any other alternative(s) to determine the voltage across 2Ω resistor?



What are they? And how?
Can you work it out by inspection?

4.2 Linearity Property

A linear circuit is one whose output (i) is linearly related (or directly proportional) to its input (v_s), as shown in Fig.

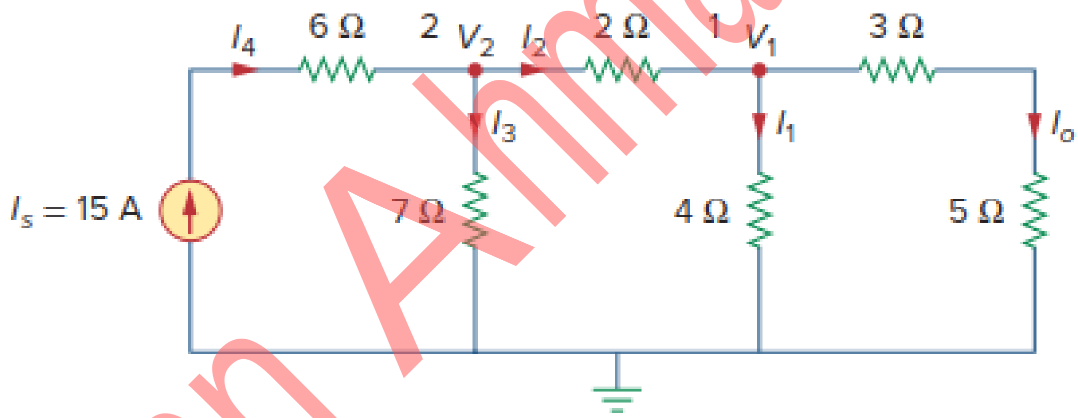


Ohm's law relates the input i to the output v,

$$v = iR \Leftrightarrow i = \frac{1}{R}v$$

Example 4.1.

By assume $I_o = 1$ A, use linearity to find the actual value of I_o in the circuit shown in Fig.



Solution:

If $I_o = 1$ A, then

$$V_1 = (3+5)I_o = 8V \quad \text{and} \quad I_1 = V_1/4 = 2 \text{ A.}$$

Applying KCL at node 1 gives

$$I_2 = I_1 + I_o = 3 \ A \Rightarrow V_2 = V_1 + 2I_2 = 8 + 6 = 14 \text{ V}, \qquad I_3 = \frac{V_2}{7} = 2\text{A}$$

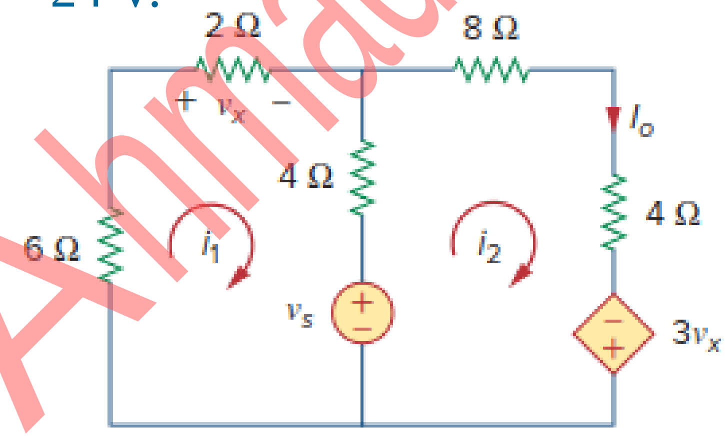
Applying KCL at node 2 gives $$I_4 = I_3 + I_2 = 5 \ \text{A}$$

Therefore, $I_s = I_4 = 5$ A.

This shows that assuming $I_o = 1$ gives $I_s = 5$ A, the actual source current of 15 A will give $I_o = 3$ A as the actual value.

Example 4.2.

For the circuit in Fig., find I_o when v_s = 12 V and v_s = 24 V.



Solution:

For loop 1,

$$6i_1 + 2i_1 + 4(i_1 - i_2) + v_s = 0$$

$$\Rightarrow \quad 12i_1 - 4i_2 + v_s = 0 \quad (1)$$

For loop 2,

$$-v_s + 4(i_2 - i_1) + 8i_2 + 4i_2 - 3v_x = 0$$

$$\Rightarrow \quad -4i_1 + 16i_2 - 3v_x - v_s = 0 \quad (2)$$

But $v_x = 2i_1$ ➔ Equation (2) becomes $-10i_1 + 16i_2 - v_s = 0 \quad (3)$

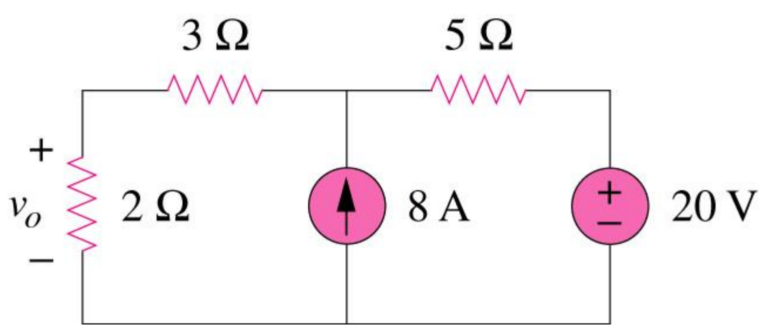
By solving Eqs. (1) and (3), we have $i_1 = -6i_2, \quad i_2 = \frac{v_s}{76}$

When, $v_s = 12\,\text{V} \Rightarrow I_o = i_2 = \frac{12}{76}\,\text{A}; \quad v_s = 24\,\text{V} \Rightarrow I_o = i_2 = \frac{24}{76}\,\text{A}$

showing that when the source value is doubled, I_o doubles.

4.3 Superposition Theorem

- It states that the voltage across (or current through) an element in a linear circuit is the algebraic sum of the voltage across (or currents through) that element due to **each** independent source acting alone.
- The principle of superposition helps us to analyze a linear circuit with more than one independent source by calculating the contribution of each independent source separately.
- For example, we consider the effects of 8A and 20V one by one, then add the two effects together for final v_o.



Steps to apply superposition principle

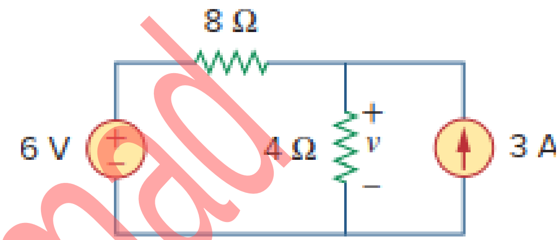
1. **Turn off** all independent sources except one source. Find the output (voltage or current) due to that active source using nodal or mesh analysis.
2. **Repeat** step 1 for each of the other independent sources.
3. **Find the total contribution (المساهمة)** by adding algebraically all the contributions due to the independent sources.

Two things have to be keep in mind:

1. When we say *turn off* all other independent sources:
   - Independent voltage sources are replaced by 0 V (short circuit) and
   - Independent current sources are replaced by 0 A (open circuit).
2. Dependent sources are left intact (تُترَك سليمة) because they are controlled by circuit variables.

Example 4.3.

Use the superposition theorem to find v in the circuit of Fig.



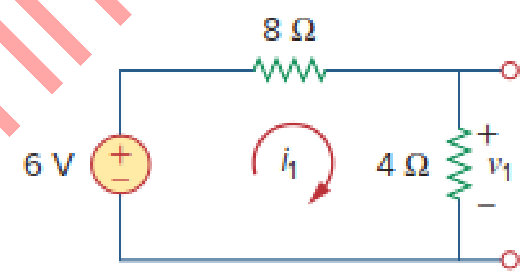
Solution:

Let v_1 and v_2 are the contributions due to the 6-V voltage source and the 3-A current source, respectively. Hence,

$$v = v_1 + v_2$$

- To obtain v_1 we set the *current source* to zero, Fig.(a). Applying KVL to the loop in Fig.(a) gives

$$12i_1 - 6 = 0 \Rightarrow i_1 = 0.5 \text{ A} \quad \Rightarrow v_1 = 4i_1 = 2 \text{ V}$$



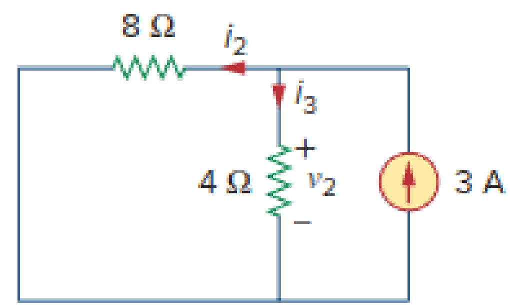
(a)

  - We may also use voltage division to get v_1 by writing

$$v_1 = \frac{4}{4+8}(6) = 2 \text{ V}$$

- To get v_2 we set the *voltage source* to zero, as in Fig.(b).
  - Using current division,

$$i_3 = \frac{8}{4+8}(3) = 2 \text{ A} \Rightarrow v_2 = 4i_3 = 8 \text{ V}$$

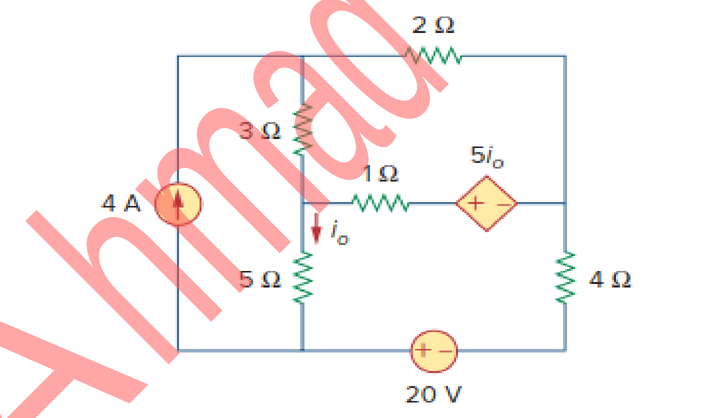


(b)

- Finally, $v = v_1 + v_2 = 2 + 8 = 10 \text{ V}$

Example 4.5.

Find i_o in the circuit of Fig. using superposition.

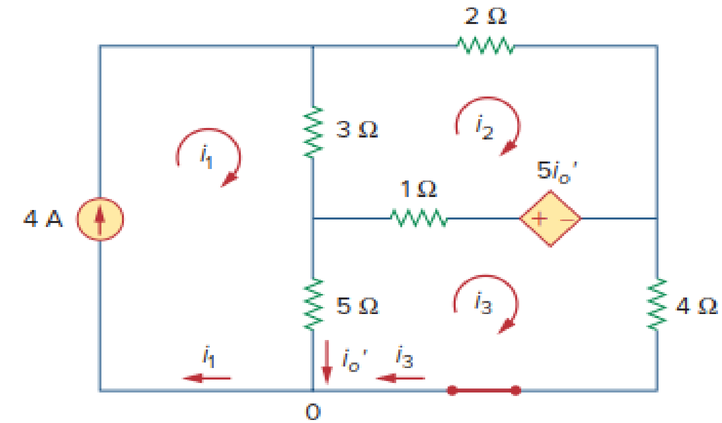


Solution:

Let $i_o^{'}$ and $i_o^{''}$ are due to the 4-A current source and 20-V voltage source respectively. Hence,

$$i_o = i_o^{'} + i_o^{''} \qquad (1)$$

To obtain $i_o^{'}$, see Fig.(a),



(a)

For loop 1,
$$i_1 = 4 \text{ A} \qquad (2)$$

For loop 2,
$$-3i_1 + 6i_2 - 1i_3 - 5i_o^{'} = 0 \qquad (3)$$

For loop 3,
$$-5i_1 - 1i_2 + 10i_3 + 5i_o^{'} = 0 \qquad (4)$$

But at node 0,
$$i_3 = i_1 - i_o^{'} = 4 - i_o^{'} \qquad (5)$$

Substituting Eqs. (2) and (5) into Eqs. (3) and (4) gives

$$3i_2 - 2i_o^{'} = 8 \qquad (6)$$

$$i_2 + 5i_o^{'} = 20 \qquad (7)$$

By solving Eqs. (6) and (7), we have

$$\boxed{i_o^{'} = \frac{52}{17} \text{ A}}$$

To obtain i_o'', see Fig.(b).

For loop 4, KVL gives

$$6i_4 - i_5 - 5i_o'' = 0 \quad (9)$$

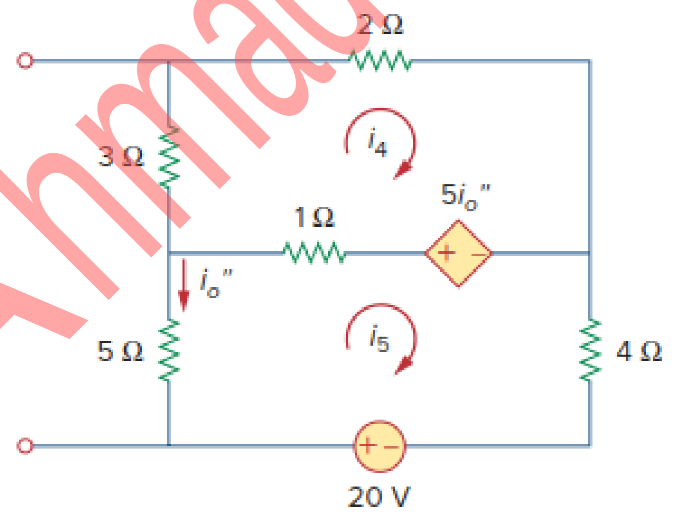
For loop 5,

$$-i_4 + 10i_5 - 20 + 5i_o'' = 0 \quad (10)$$

But $i_5 = -i_o''$. Substituting this in Eqs. (9) and (10) gives

$$6i_4 - 4i_o'' = 0 \quad (11)$$

$$i_4 + 5i_o'' = -20 \quad (12)$$



(b)

By solving Eqs. (11) and (12), we have:

$$i_o'' = -\frac{60}{17}\,\text{A}$$

Finally,

$$i_o = i_o' + i_o'' = \frac{52}{17} - \frac{60}{17} = -\frac{8}{17} = -0.4706\,\text{A}$$

4.4 Source Transformation

- **Source transformation** is another tool for simplifying circuits. Basic to these tools is the concept of **equivalence**.

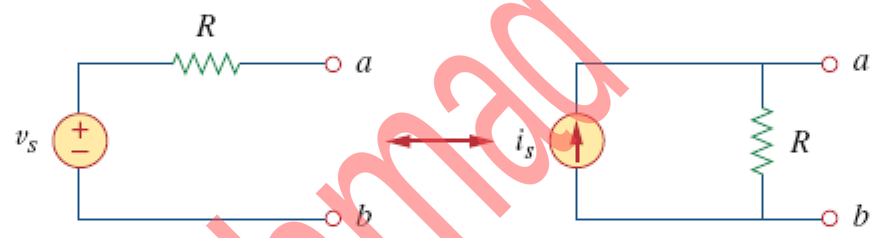
- An equivalent circuit is one whose *v-i* characteristics are identical with the original circuit.

- It is the process of replacing a voltage source v_S in series with a resistor R by a current source i_S in parallel with a resistor R, or vice versa.

- Source transformation requires that $v_s = i_s R$ or $i_s = \frac{v_s}{R}$

Note from Figures:

- The arrow of the current source is directed toward the positive terminal of the voltage source.

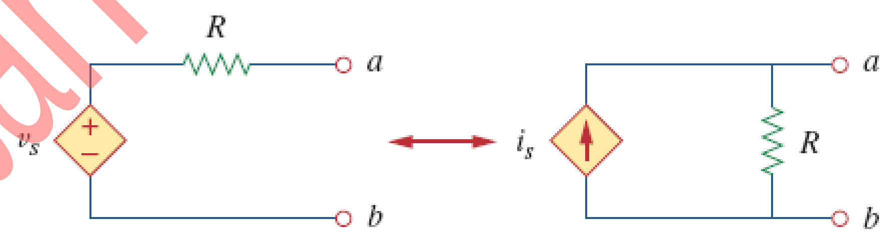


Transformation of independent sources

$$v_s = i_s R \quad \text{or} \quad i_s = \frac{v_s}{R}$$

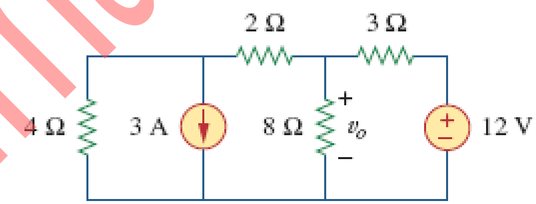
Note from eq.

- The source transformation is not possible when R = 0 for voltage source and R = ∞ for current source.



Transformation of dependent sources

Example 4.6. Use source transformation to find v_o in the circuit of Fig.



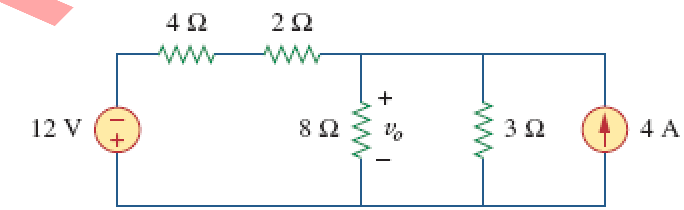
Solution:

- We first transform the current and voltage sources to obtain the circuit in Fig.(a).
- Combining the 4-Ω and 2-Ω resistors in series and transforming the 12-V voltage source gives us Fig.(b).
- We now combine the 3-Ω and 6-Ω resistors in parallel to get 2-Ω.
- We also combine the 2-A and 4-A current sources to get a 2-A source.
- Thus, by repeatedly applying source transformations, we obtain the circuit in Fig.(c).
- We use current division in Fig.(c) to get

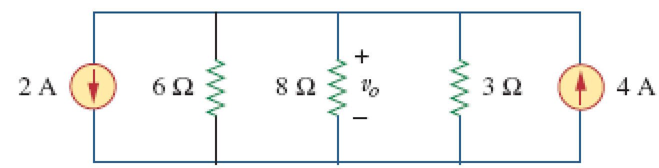
$$i = \frac{2}{2+8}(2) = 0.4 \text{ A} \Rightarrow v_o = 8i = 8(0.4) = 3.2 \text{ V}$$

- Alternatively, since the 8-Ω and 2-Ω and resistors in Fig.(c) are in parallel, they have the same voltage across them. Hence,

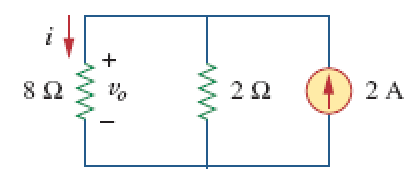
$$v_o = (8 \| 2)(2 \text{ A}) = \frac{8 \times 2}{8+2}(2) = 3.2 \text{ V}$$



(a)



(b)



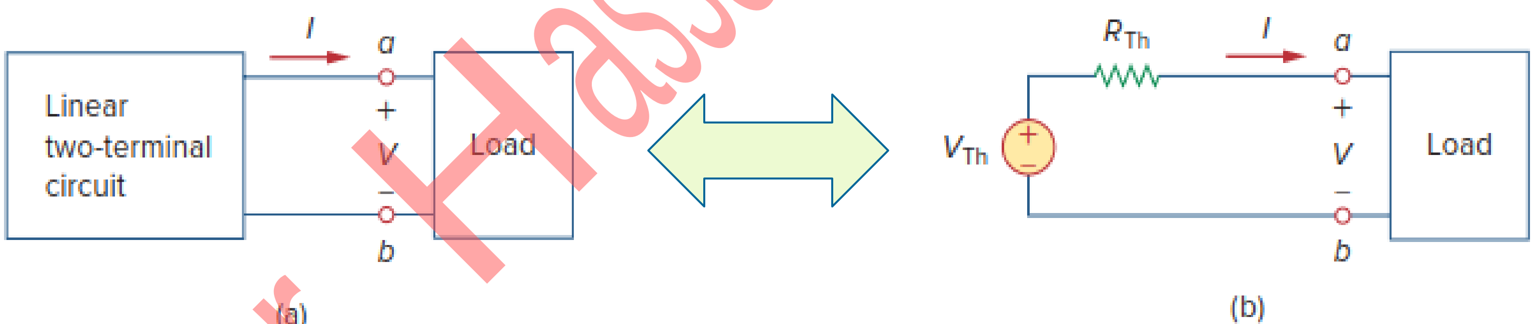
(c)

4.5 Thevenin's Theorem

Thevenin's theorem states that a linear two-terminal circuit, Fig.(a), can be replaced by an equivalent circuit, Fig.(b), consisting of a voltage source V_{Th} in series with a resistor R_{Th},

where

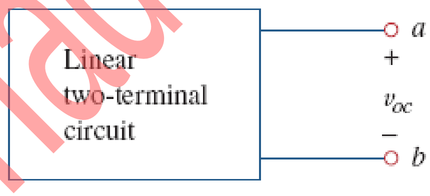
- V_{Th} is the open-circuit voltage at the terminals.
- R_{Th} is the input or equivalent resistance at the terminals when the independent sources are turned off.



Our major concern right now is how to **find** the Thevenin equivalent voltage V_{Th} and resistance R_{Th} .

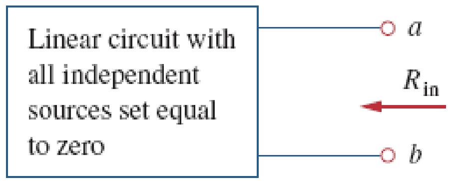
V_{Th}

- By removing the load, the terminals are made *open-circuited*, then no current flows, so that the open-circuit voltage across the terminals, Fig.(a) must be equal to the voltage source in Fig. (b), (*see previous slide*).
- Thus, V_{Th} is the ***open-circuit voltage*** across the terminals, Fig.(a); that is, $$V_{Th} = v_{oc}$$



$V_{Th} = v_{oc}$

(a)



$R_{Th} = R_{in}$

(b)

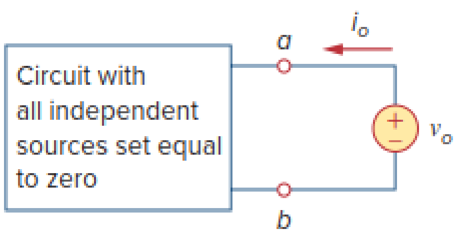
R_{Th}

- Again, with the *load disconnected* and terminals *a-b* *open-circuited*, we turn off all independent sources.
- The input resistance (or equivalent resistance) of the dead circuit at the terminals *a-b* must be equal to R_{Th}, Fig.(b), that is, $$R_{Th} = R_{in} = R_{eq}$$

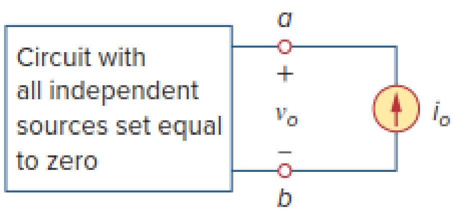
NOTE.

If the network has dependent sources, we turn of f all independent sources and leave the dependent source alone.

We may insert a voltage (v_o) or current source (i_o) at terminals a-b as shown in Fig.



$R_{Th} = \frac{v_o}{i_o}$

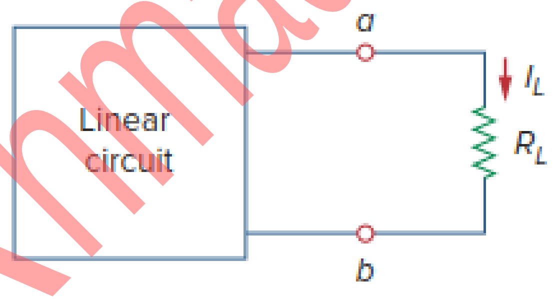


$R_{Th} = \frac{v_o}{i_o}$

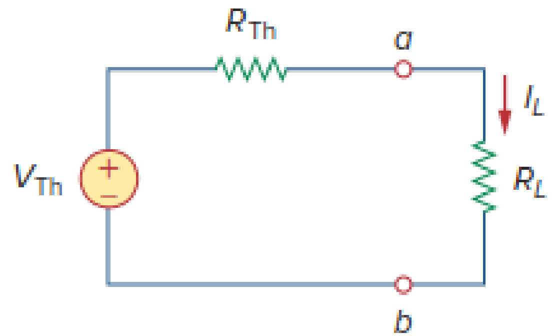
- A linear circuit with a variable load can be replaced by the Thevenin equivalent, exclusive of the load.
- The equivalent network behaves the same way externally as the original circuit.

$$I_L = \frac{V_{Th}}{R_{Th} + R_L};$$

$$V_L = R_L I_L = \frac{R_L}{R_{Th} + R_L} V_{Th}$$



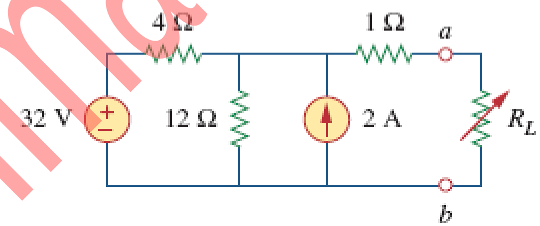
(a)



(b)

Example 4.7.

Find the Thevenin equivalent circuit of the circuit shown in Fig., to the left of the terminals *a*-*b*. Then find the current through R_L= 6, 16 Ω.

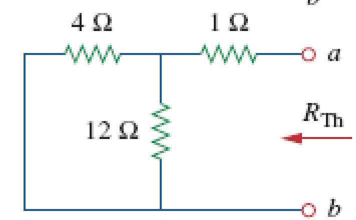


Solution:

R_{Th}

- We find by turning off the 32-V voltage source (replacing it with a short circuit) and the 2-A current source (replacing it with an open circuit). The circuit becomes what is shown in Fig.(a).
- Thus,

$$R_{Th} = (4 \| 12) + 1 = \frac{4 \times 12}{4+12} + 1 = 4\Omega$$

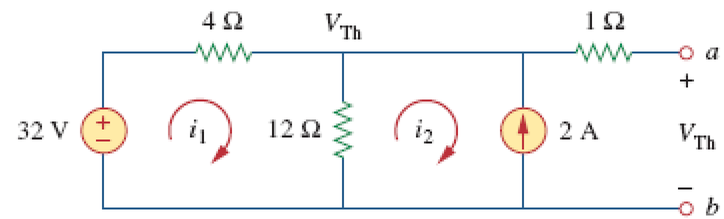


(a)

V_{Th}

- To find consider the circuit in Fig.(b).
- Applying mesh analysis to the two loops, we obtain

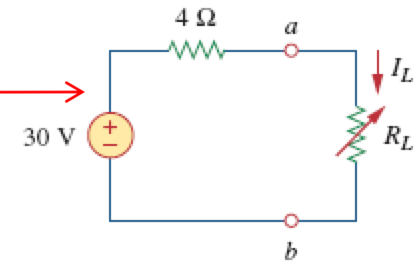
$$-32 + 4i_1 + 12(i_1 - i_2) = 0, \qquad i_2 = -2\text{A}$$



(b)

- Solving for i_1, we get: $i_1 = 0.5$ A.
- Thus, $V_{Th} = 12(i_1 - i_2) = 30$ V

Thevenin equivalent circuit

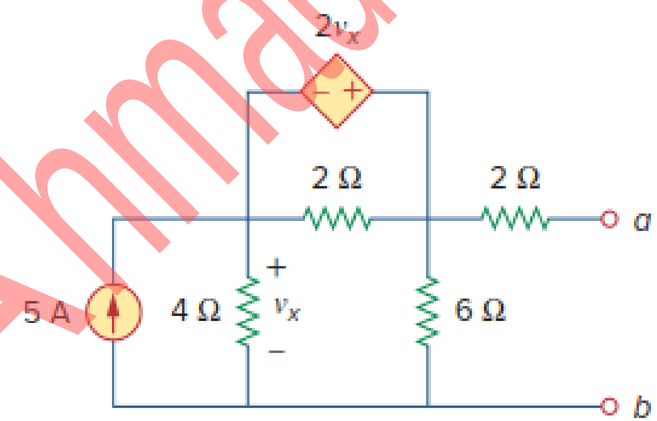


(c)

$$I_L = \frac{V_{Th}}{R_{Th} + R_L} = \frac{30}{4 + R_L} \Rightarrow I_{L_{(6\Omega)}} = 3\text{A}, \quad I_{L_{(16\Omega)}} = 1.5\text{A}$$

Example 4.8.

Find the Thevenin equivalent of the circuit in Fig. at terminals *a-b*.



Solution:

To find $\boldsymbol{R_{Th}}$, we excite the network with a voltage source v_o = 1 V connected to the terminals as indicated in Fig. (a).

Our goal is to find the current i_o through the terminals, and then obtain $R_{Th} = v_o / i_o = 1/i_o$.

(Alternatively, we may insert a 1-A current source, find the corresponding voltage v_o, and obtain $R_{Th} = v_o/1$).

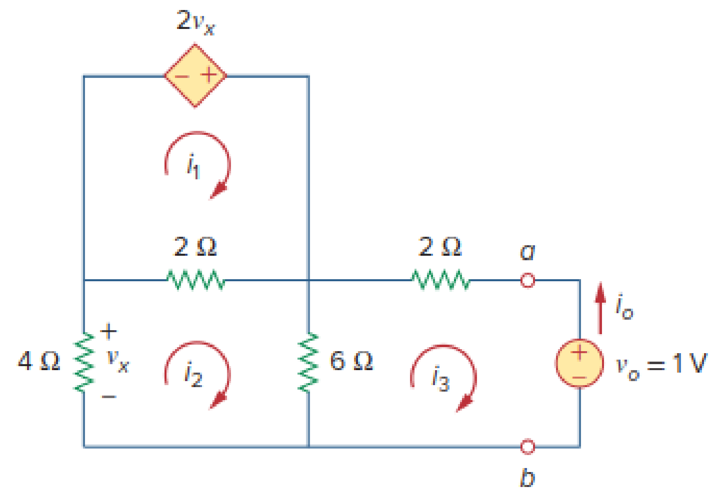
For loop 1, $-2v_x + 2(i_1 - i_2) = 0 \Rightarrow v_x = i_1 - i_2$

But $-4i_2 = v_x = i_1 - i_2;$

hence, $i_1 = -3i_2$ (1)

For loop 2 and 3, $4i_2 + 2(i_2 - i_1) + 6(i_2 - i_3) = 0$ (2)

$6(i_3 - i_2) + 2i_3 + 1 = 0$ (3)

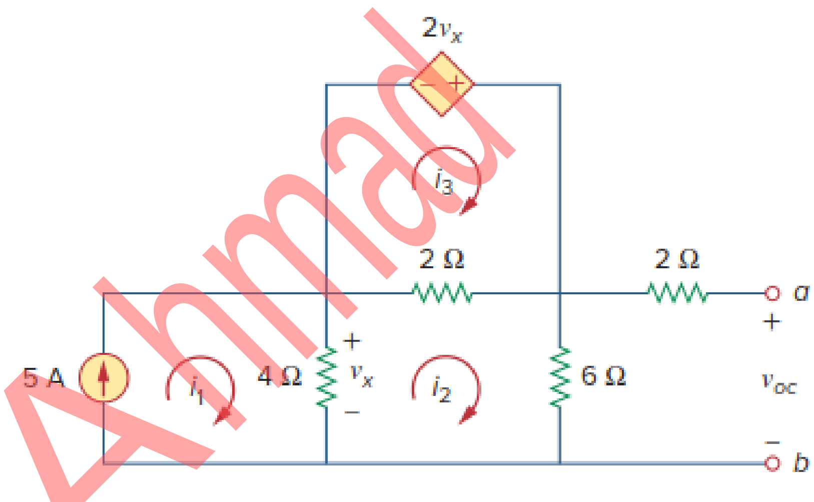


(a)

Solving these equations gives $i_3 = -\frac{1}{6}$ A

But $i_o = -i_3 = 1/6$ A. Hence, $R_{Th} = 1\text{V}/i_o = 6\Omega.$

To get $\boldsymbol{V_{\text{Th}}}$, we find v_{oc} in the circuit of Fig.(b). Applying mesh analysis, we get



(b)

For loop 1, $i_1 = 5$ (4)

For loop 2,

$$-2v_x + 2(i_3 - i_2) = 0 \Rightarrow v_x = i_3 - i_2 \quad (5)$$

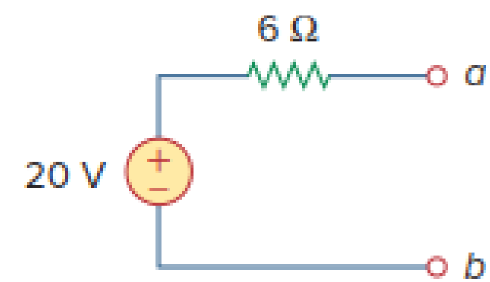
For loop 3, $4(i_2 - i_1) + 2(i_2 - i_3) + 6i_2 = 0$

$$\Rightarrow \quad 12i_2 - 4i_1 - 2i_3 = 0 \quad (6)$$

But $4(i_1 - i_2) = v_x$ (7)

Solving the equations (5), (6) and (7) we have $i_2 = \dfrac{10}{3}\text{A}$

Hence, $\boxed{V_{\text{Th}} = v_{oc} = 6i_2 = 20\text{ V}}$



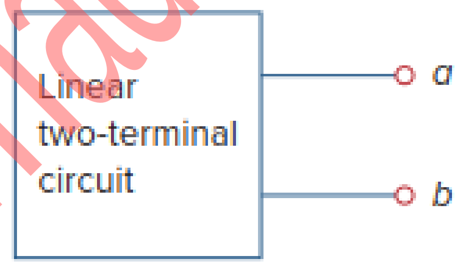
The Thevenin equivalent is as shown in Fig.

4.6 Norton's Theorem

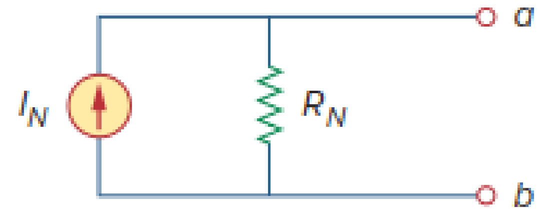
- **Norton's theorem** states that a linear two-terminal circuit can be replaced by an equivalent circuit of a current source I_N in parallel with a resistor R_N,

where

  - I_N is the short circuit current (i_{sc}) through the terminals.
  - R_N is the input or equivalent resistance at the terminals when the independent sources are turned off.



(a)



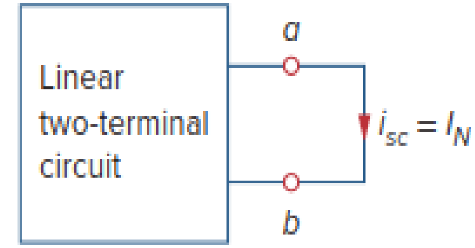
(b)

- Thevenin and Norton resistances are equal; that is,

$$R_N = R_{\text{Th}}$$

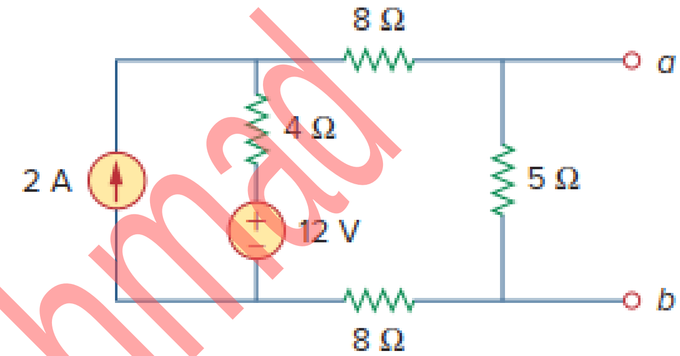
- The Thevenin's and Norton equivalent circuits are related by a source transformation; that is,

$$I_N = \frac{V_{\text{Th}}}{R_{\text{Th}}}$$



Example 4.9.

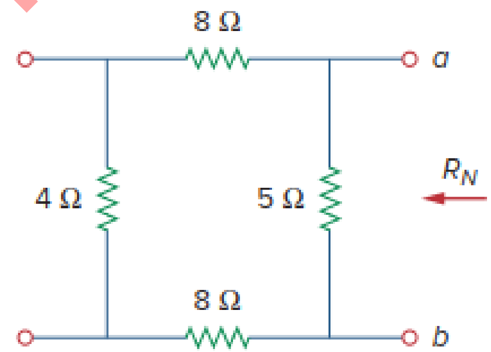
Find the Norton equivalent circuit of the circuit in Fig. at terminals *a-b*.



Solution:

To find R_N, we set the independent sources equal to zero. See Fig.(a). Thus,

$$R_N = 5 \,||\, (8+4+8) = \frac{20 \times 5}{20+5} = 4\Omega$$



(a)

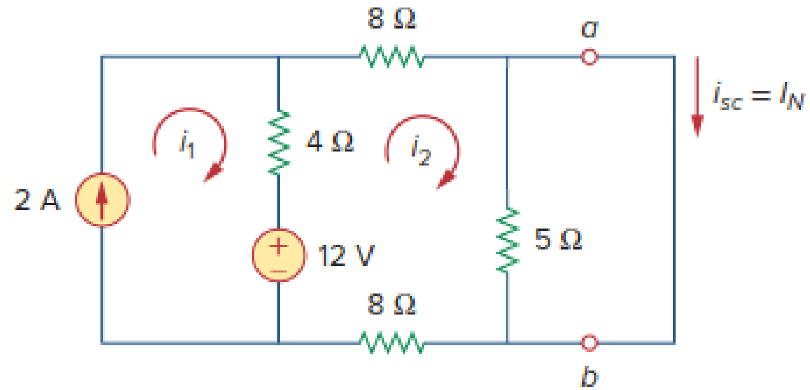
To find I_N, we short-circuit terminals *a* and *b*, Fig.(b).

We ignore the 5-Ω resistor because it has been short-circuited. Applying mesh analysis, we obtain

$$i_1 = 2\,\text{A}, \quad 20 i_2 - 4 i_1 - 12 = 0$$

From these equations, we obtain

$$i_2 = 1\text{A} = i_{sc} = I_N$$



(b)

Alternatively, we may determine I_N from V_{Th}/R_{Th}. We obtain V_{Th} as the open-circuit voltage across terminals *a* and *b* in Fig.(c).

Using mesh analysis, we obtain

$$i_3 = 2\,\text{A}$$

$$25i_4 - 4i_3 - 12 = 0 \Rightarrow i_4 = 0.8\,\text{A}$$

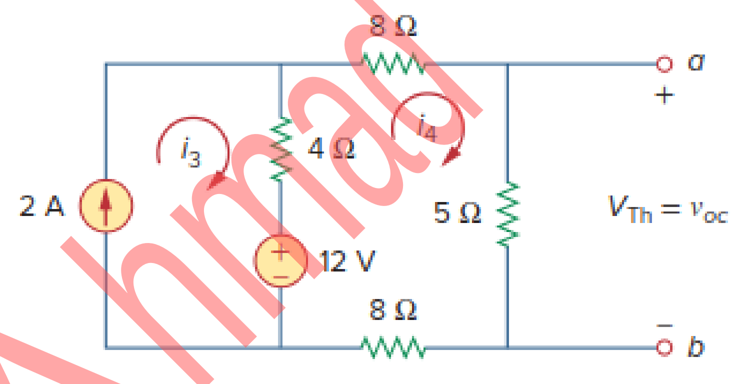
and

$$v_{oc} = V_{Th} = 5i_4 = 4\,\text{V}$$

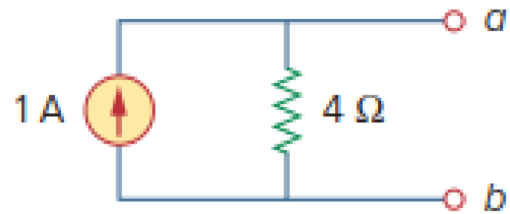
Hence,

$$I_N = \frac{V_{Th}}{R_{Th}} = \frac{4}{4} = 1\,\text{A}$$

Thus, the Norton equivalent circuit is as shown in Fig.

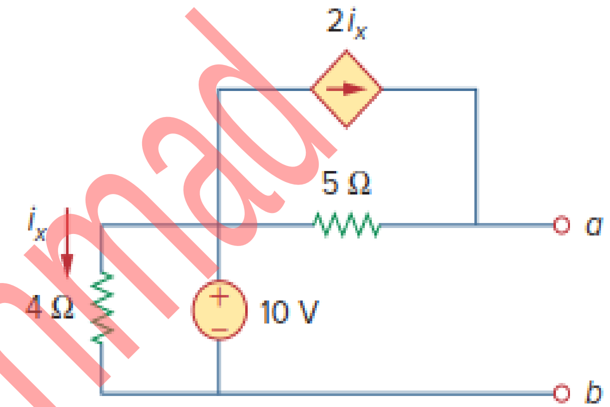


(c)



Example 4.10.

Using Norton's theorem, find $\boldsymbol{R_N}$ and $\boldsymbol{I_N}$ of the circuit in Fig. at terminals *a*-*b*.



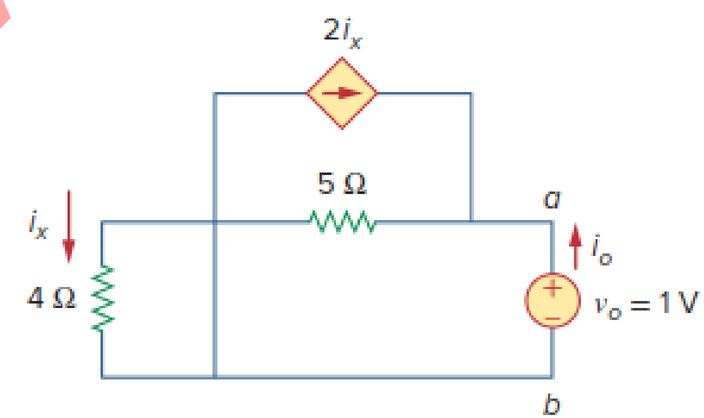
Solution:

To find $\boldsymbol{R_N}$, we set the independent voltage source equal to zero and connect a voltage source of $\boldsymbol{v_o}$ = 1 V, Fig.(a).

In Fig.(a), the 4-Ω resistor is short-circuited, so we ignore it.

The 5- Ω resistor, the voltage source, and the dependent current source are all in parallel.

Hence, $i_x = 0$.



(a)

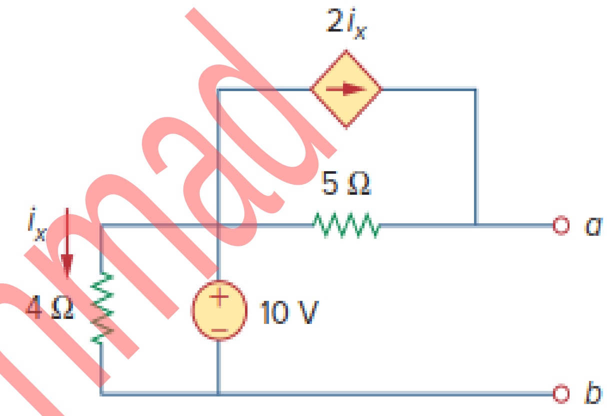
At nod *a*,

$$i_o = \frac{1\text{V}}{5\Omega} = 0.2\,\text{A} \Rightarrow$$

$$R_N = \frac{v_o}{i_o} = 5\Omega$$

To find I_N, we short-circuit terminals *a* and *b* and find the current i_{sc}, as in Fig. (b).

Note from this figure that the 4 Ω resistor, the 10-V voltage source, the 5-Ω resistor, and the dependent current source are all in parallel.



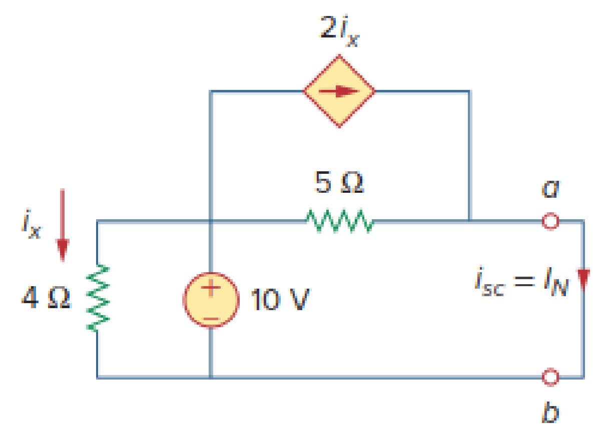
Hence,

$$i_x = \frac{10}{4} = 2.5\,\text{A}$$

At nod *a*, KCL gives

$$i_{sc} = \frac{10}{5} + 2i_x = 2 + 2(2.5) = 7\,\text{A}$$

Thus, $\boxed{I_N = 7\,\text{A}}$

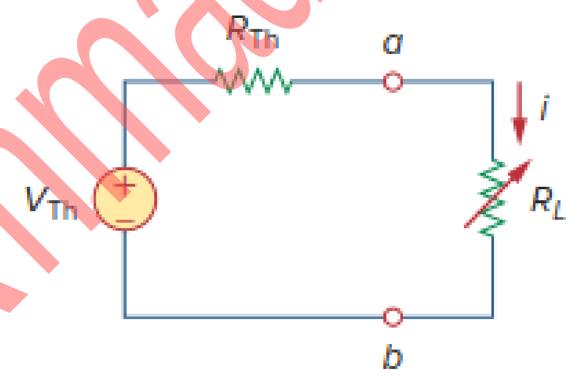


(b)

4.7 Maximum Power Transfer

- If the entire circuit is replaced by its Thevenin equivalent **except** for the **load**, the **power delivered** to the load is:

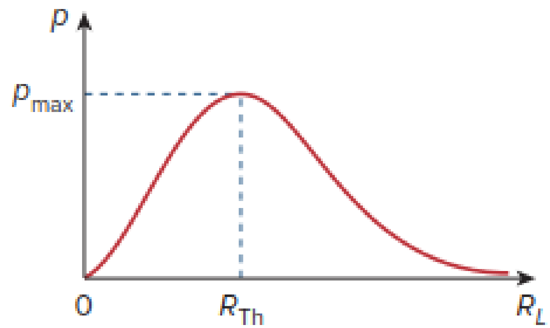
$$p = i^2 R_L = \left(\frac{V_{Th}}{R_{Th} + R_L} \right)^2 R_L, \quad \text{with } R_L \neq R_{Th}$$



- Maximum power is transferred to the load when

$$R_L = R_{Th}$$

  - This is known as the **maximum power theorem**.



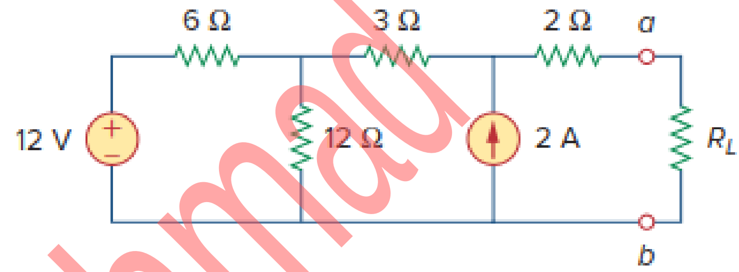
- Hence,

$$p_{max} = \frac{V_{Th}^2}{4R_{Th}}$$

Example 4.11.

Find the value of R_L for maximum power transfer in the circuit of Fig.

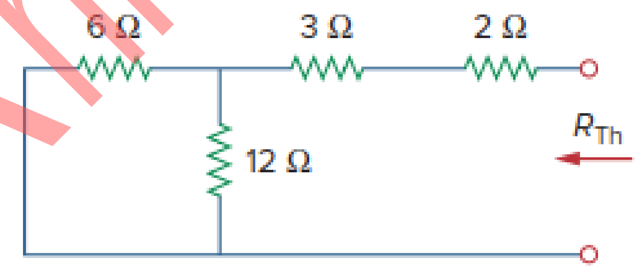
Find the maximum power.



Solution:

To get R_{Th}, we use the circuit in Fig.(a) and obtain

$$\boxed{R_{\text{Th}} = 2 + 3 + 6 \,||\, 12 = 9\,\Omega}$$

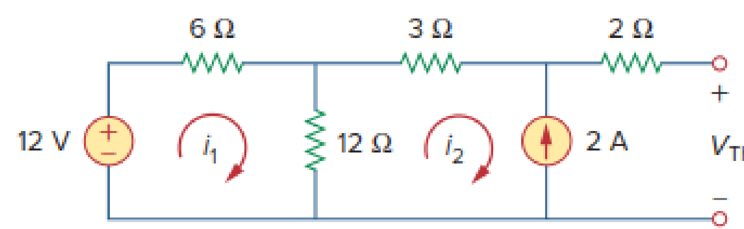


(a)

To get V_{Th}, we applying mesh analysis to the circuit in Fig.(b).

$$-12 + 18i_1 - 12i_2 = 0, \quad i_2 = -2\,\text{A}$$

$$\Rightarrow i_1 = -2/3\,\text{A}$$



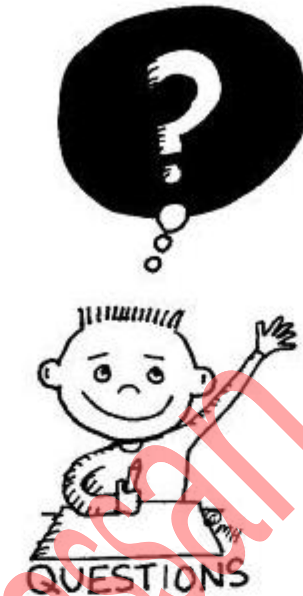
(b)

Applying KVL around the outer loop to get V_{Th} across terminals *a-b*, we obtain

$$-12 + 6i_1 + 3i_2 + 2(0) + V_{\text{Th}} = 0 \Rightarrow \boxed{V_{\text{Th}} = 22\,\text{V}}$$

For maximum power transfer, and the maximum power is

$$R_L = R_{Th} = 9\,\Omega \Rightarrow p_{\max} = \frac{V_{\text{Th}}^2}{4R_{\text{Th}}} = \frac{22^2}{2 \times 9} = 13.44\,\text{W}$$



The end of chapter 4